Thermoelastic behaviour of masonry-like solids with temperature-dependent Young's modulus

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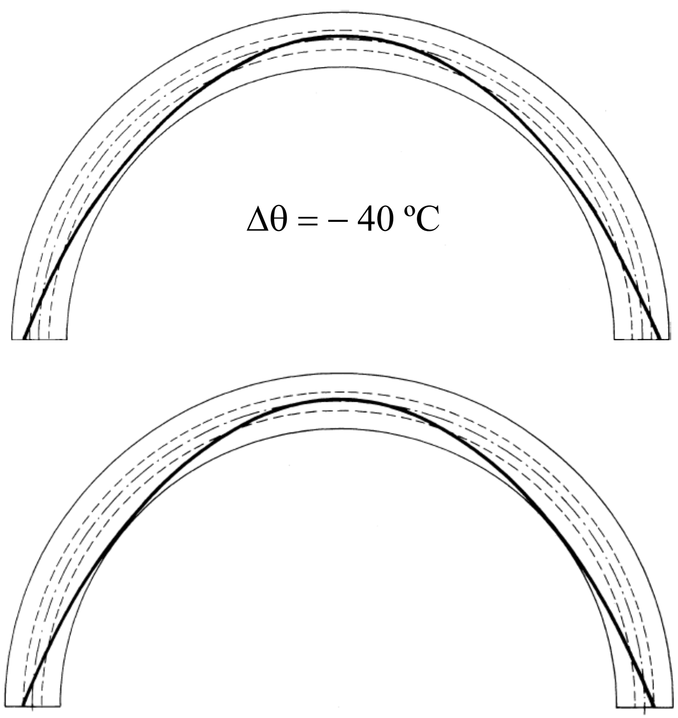
Masonry-like solids in the presence of thermal variations

Masonry-like (no-tension) materials are nonlinear elastic materials whose constitutive equation is adopted to model the mechanical behaviour of solids that do not withstand tensile stresses, such as masonry and stones.

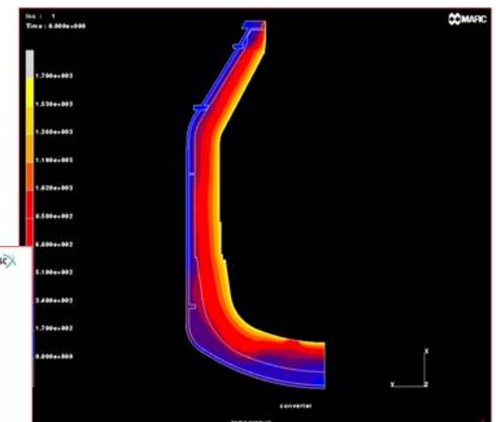
There are many engineering problems in which the presence of thermal variations (and then thermal dilatation) must be taken into account

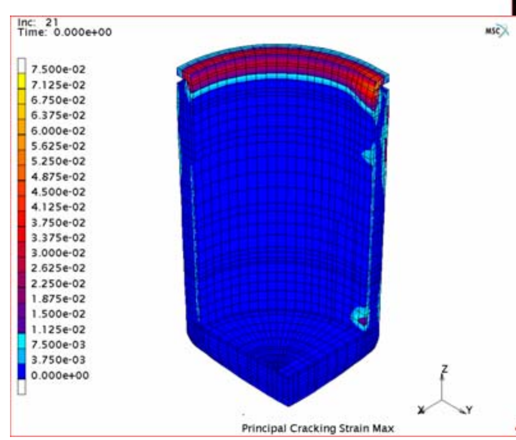
Masonry constructions subjected to seasonal thermal variations



Refractory linings of converters and ladles employed in the iron and steel industry

Δθ ≤ 1600 ºC





The thermal variation is so high that the dependence of material constants on temperature cannot be ignored

Masonry-like materials under non-isothermal conditions

Sym the space of symmetric tensors

Sym^{+} the set of positive semidefinite symmetric tensors

Sym^{-} the set of negative semidefinite symmetric tensors

$\theta \in [\theta_1, \theta_2]$ the absolute temperature, $\theta_0 \in [\theta_1, \theta_2]$ the reference temperature

$\mathbf{E} \in$ Sym, the symmetric part of the displacement gradient, $\mathbf{I}$ the identity tensor

$\mathbf{T} \in$ Sym, the Cauchy stress tensor

$\beta(\theta)$ the thermal expansion $\beta(\theta_0) = 0$

$\beta(\theta)\mathbf{I}$, the thermal dilatation due to the the thermal variation $\theta - \theta_0$

$E(\theta) > 0$, the Young's modulus, $0 \leq \nu(\theta) < 1/2$ the Poisson's ratio

No limitations on $\theta - \theta_0$	$\mathbf{E} - \beta(\theta)\mathbf{I} = O(\delta)$	(is small)

For $(\mathbf{E}, \theta) \in \mathrm{Sym} \times [\theta_1, \theta_2]$ there exists a unique triple $(\mathbf{T}, \mathbf{E}^e, \mathbf{E}^f)$ of elements of Sym such that

$$\begin{cases} \mathbf{E} - \beta(\theta)\mathbf{I} = \mathbf{E}^e + \mathbf{E}^f \\ \mathbf{T} = \dfrac{E(\theta)}{1+\nu(\theta)}\left[\mathbf{E}^e + \dfrac{\nu(\theta)}{1-2\nu(\theta)} tr(\mathbf{E}^e)\mathbf{I}\right] \\ \mathbf{T} \in \mathrm{Sym}^-, \quad \mathbf{E}^f \in \mathrm{Sym}^+ \\ \mathbf{T} \cdot \mathbf{E}^f = 0 \end{cases} \tag{1}$$

The nonlinear elastic material with stress function $\widehat{\mathbf{T}}(\mathbf{E}, \theta) = \mathbf{T}$ is called **masonry-like material**

$\mathbf{E}^e$ and $\mathbf{E}^f$ are called the **elastic** and **fracture** parts of $\mathbf{E} - \beta(\theta)\mathbf{I}$

Lucchesi M., Padovani C., Pasquinelli G., 2000. Thermodynamics of no-tension materials. International Journal of Solids and Structures 37, pp. 6581-6004.

Lucchesi M., Padovani C., Pasquinelli G., Zani N., 2008. Masonry constructions: mechanical models and numerical applications. Lecture Notes in Applied and Computational Mechanics vol. 39, Springer-Verlag Berlin Heidelberg.

In the absence of thermal variations we get the constitutive equation of masonry-like materials introduced by Heyman and Di Pasquale and Del Piero in the 80s

The equations of the thermoelasticity of masonry-like materials are coupled

If we assume that

$$\mathbf{E} = O(\delta), \quad \beta(\theta) = O(\delta), \quad \beta'(\theta) = O(\delta), \quad \dot{\mathbf{E}} = O(\delta), \quad \dot{\theta} = O(\delta)$$

then the equations of thermoelasticity are **uncoupled** and can be integrated separately.

The uncoupled equilibrium problem of masonry-like solids with temperature dependent material properties subjected to thermal loads is solved via the finite element code **NOSA** developed by the Mechanics of Materials and Structures Laboratory for nonlinear structural analysis

Padovani C., Pasquinelli G., Zani N., 2000. A numerical method for solving equilibrium problems of no-tension solids in the presence of thermal expansion. Comput. Methods Appl. Mech. Engrg. 90, pp. 55-73.

Lucchesi M., Padovani C., Pasquinelli G., Zani N., 2008. Masonry constructions: mechanical models and numerical applications. Lecture Notes in Applied and Computational Mechanics vol. 39, Springer-Verlag Berlin Heidelberg.

Spherical container made of a masonry-like material subjected to thermal loads

Let us consider a spherical container *S* with inner radius *a* and outer radius *b*, made of a masonry-like material in the absence of body forces, subjected to surface tractions and temperatures such that the problem has spherical symmetry

For r, ϑ and φ, with $r \geq 0$, $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ the spherical coordinates

$\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\varphi$ the corresponding unit vectors and $\otimes$ the tensor product

The displacement vector and the infinitesimal strain tensor are

$$\mathbf{u} = u\mathbf{e}_r, \quad \mathbf{E} = \varepsilon_r \mathbf{e}_r \otimes \mathbf{e}_r + \varepsilon_\varphi(\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi)$$

with

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\varphi = \frac{u}{r}$$

The stress tensor and the fracture strain are

$$\mathbf{T} = \sigma_r \mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\varphi(\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi)$$

$$\mathbf{E}^f = \varepsilon_r^f \mathbf{e}_r \otimes \mathbf{e}_r + \varepsilon_\varphi^f(\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi).$$

The solution to the constitutive equation (1) in the case of spherical symmetry is given by

$$\sigma_r = \frac{E(\theta)}{1+\nu(\theta)}\{\varepsilon_r - \beta(\theta) + \frac{\nu(\theta)}{1-2\nu(\theta)}[\varepsilon_r + 2\varepsilon_\varphi - 3\beta(\theta)]\},$$

$$\sigma_\varphi = \frac{E(\theta)}{1+\nu(\theta)}\{\varepsilon_\varphi - \beta(\theta) + \frac{\nu(\theta)}{1-2\nu(\theta)}[\varepsilon_r + 2\varepsilon_\varphi - 3\beta(\theta)]\},$$

$$\varepsilon_r^{\mathrm{f}} = 0, \quad \varepsilon_\varphi^{\mathrm{f}} = 0;$$

If $(\varepsilon_r, \varepsilon_\varphi, \theta)$ belongs to

$$\mathcal{S}_1 = \{(\varepsilon_r, \varepsilon_\varphi, \theta) |\ \varepsilon_\varphi + \nu(\theta)\varepsilon_r - [1+\nu(\theta)]\beta(\theta) \le 0,$$
$$2\nu(\theta)\varepsilon_\varphi + (1-\nu(\theta))\varepsilon_r - [1+\nu(\theta)]\beta(\theta) \le 0\},$$

$$\sigma_r = 0, \quad \sigma_\varphi = 0,$$

$$\varepsilon_r^{\mathrm{f}} = \varepsilon_r - \beta(\theta), \quad \varepsilon_\varphi^{\mathrm{f}} = \varepsilon_\varphi - \beta(\theta);$$

If $(\varepsilon_r, \varepsilon_\varphi, \theta)$ belongs to

$$\mathcal{S}_2 = \{(\varepsilon_r, \varepsilon_\varphi, \theta) |\ \varepsilon_r - \beta(\theta) \ge 0,\ \varepsilon_\varphi - \beta(\theta) \ge 0\}$$

$$\sigma_r = E(\theta)[\varepsilon_r - \beta(\theta)], \quad \sigma_\varphi = 0,$$

$$\varepsilon_r^{\mathrm{f}} = 0, \quad \varepsilon_\varphi^{\mathrm{f}} = \nu(\theta)\varepsilon_r + \varepsilon_\varphi - (1+\nu(\theta))\beta(\theta);$$

If $(\varepsilon_r, \varepsilon_\varphi, \theta)$ belongs to

$$\mathcal{S}_3 = \{(\varepsilon_r, \varepsilon_\varphi, \theta) | \varepsilon_r \le \varepsilon_\varphi, \varepsilon_r - \beta(\theta) \le 0,$$
$$\varepsilon_\varphi + \nu(\theta)\varepsilon_r - [1+\nu(\theta)]\beta(\theta) \ge 0\}$$

$$\sigma_r = 0, \quad \sigma_\varphi = \frac{E(\theta)}{1-\nu(\theta)}[\varepsilon_\varphi - \beta(\theta)],$$

$$\varepsilon_r^{\mathrm{f}} = \varepsilon_r + \frac{2\nu(\theta)}{1-\nu(\theta)}\varepsilon_\varphi - \frac{1+\nu(\theta)}{1-\nu(\theta)}\beta(\theta), \quad \varepsilon_\varphi = 0.$$

If $(\varepsilon_r, \varepsilon_\varphi, \theta)$ belongs to

$$\mathcal{S}_4 = \{(\varepsilon_r, \varepsilon_\varphi, \theta) | \varepsilon_r \ge \varepsilon_\varphi, \varepsilon_\varphi - \beta(\theta) \le 0,$$
$$2\nu(\theta)\varepsilon_\varphi + [1-\nu(\theta)]\varepsilon_r - [1+\nu(\theta)]\beta(\theta) \ge 0\}$$

The equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\varphi) = 0$$

with the boundary conditions

$$\sigma_r(a) = -p_a, \quad \sigma_r(b) = -p_b,$$

where p_a and p_b are given positive quantities

For $\theta(a) = \theta_a, \quad \theta(b) = \theta_b$ with $\theta_a \geq \theta_b.$ we consider the steady temperature distribution

$$\theta(r) = \frac{ab(\theta_a - \theta_b)}{b-a}\frac{1}{r} + \frac{b\theta_b - a\theta_a}{b-a}, \qquad r \in [a, b]$$

we assume that the thermal expansion is $\beta(\theta) = \alpha(\theta - \theta_0)$ with α positive constant

We want to compare the solutions to the equilibrium problem corresponding to different choices of the Young's modulus (with ν=0)

If the Young's modulus does not depend on temperature, then the solution can be calculated explicitly

If the Young's modulus depends on temperature, then the solution can be calculated numerically via the finite element code NOSA

$$E_1 = 5000 \text{ MPa},$$
$$E_2 = 3500 \text{ MPa},$$
$$E_3 = 2500 \text{ MPa},$$
$$E_V(\theta) = \frac{4.95\times 10^{12}}{\theta+676.85}, \quad \theta \in [\theta_1, \theta_2].$$

$$a = 1 \text{ m}$$
$$b = 2 \text{ m}$$
$$\nu = 0$$
$$p_a = 1.5 \times 10^6 \text{ Pa}$$
$$p_b = 1.0 \times 10^6 \text{ Pa}$$
$$\alpha = 10^{-5} \text{ K}^{-1}$$
$$\theta_b = 40\ {}^\circ\text{C}$$
$$\theta_0 = 30\ {}^\circ\text{C}$$

$$\theta_a = 1830\ {}^\circ\text{C} \ (\text{case } 1)$$
$$\theta_a = 1030\ {}^\circ\text{C} \ (\text{case } 2)$$
$$\theta_a = 530\ {}^\circ\text{C} \ (\text{case } 3)$$
$$\theta_a = 230\ {}^\circ\text{C} \ (\text{case } 4)$$
$$\theta_a = 130\ {}^\circ\text{C} \ (\text{case } 5)$$

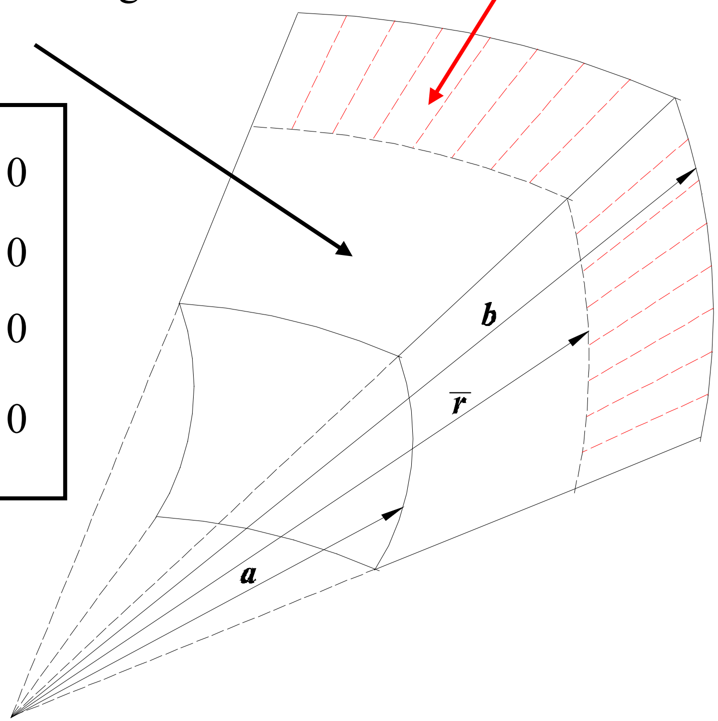
$\overline{r}$ is the transition radius which separates the entirely compressed region from the cracked region

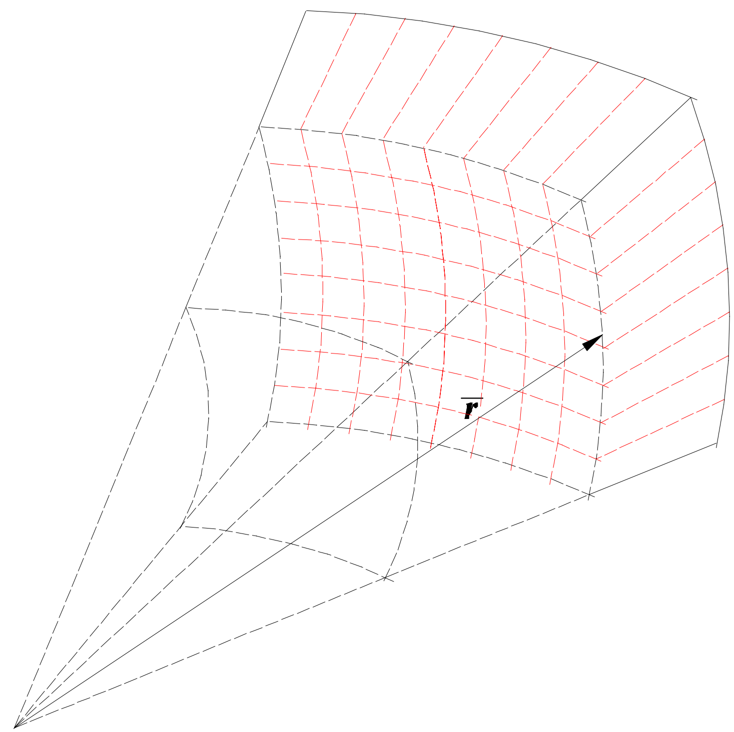
Cracked region

$$\sigma_r < 0 \qquad \varepsilon_r^{\mathrm{f}} = 0$$
$$\sigma_\varphi = 0 \qquad \varepsilon_\varphi^{\mathrm{f}} > 0$$

compressed region

$$\sigma_r < 0$$
$$\sigma_\varphi < 0$$
$$\varepsilon_r^{\mathrm{f}} = 0$$
$$\varepsilon_\varphi^{\mathrm{f}} = 0$$





$$\sigma_r(r) = \frac{a^3\overline{r}^3}{\overline{r}^3 - a^3}[\alpha E(\theta_a - \overline{\theta}) - p_a + \overline{p}]\frac{1}{r^3} - \frac{a\overline{r}}{\overline{r} - a}\alpha E(\theta_a - \overline{\theta})\frac{1}{r}$$

$$+\frac{1}{\overline{r}^3 - a^3}[p_a a^3 - \overline{p}^3\overline{r} + \alpha E a\overline{r}(a + \overline{r})(\theta_a - \overline{\theta})], \qquad r \in [a, \overline{r}],$$

$$\sigma_r(r) = -p_b\frac{b^2}{r^2}, \qquad r \in [\overline{r}, b],$$

$$\sigma_\varphi(r) = -\frac{a^3\overline{r}^3}{2(\overline{r}^3 - a^3)}[\alpha E(\theta_a - \overline{\theta}) - p_a + \overline{p}]\frac{1}{r^3} - \frac{a\overline{r}}{2(\overline{r} - a)}\alpha E(\theta_a - \overline{\theta})\frac{1}{r}$$

$$+\frac{1}{\overline{r}^3 - a^3}[p_a a^3 - \overline{p}^3\overline{r} + \alpha E a\overline{r}(a + \overline{r})(\theta_a - \overline{\theta})], \qquad r \in [a, \overline{r}],$$

$$\sigma_\varphi(r) = 0, \qquad r \in [\overline{r}, b],$$

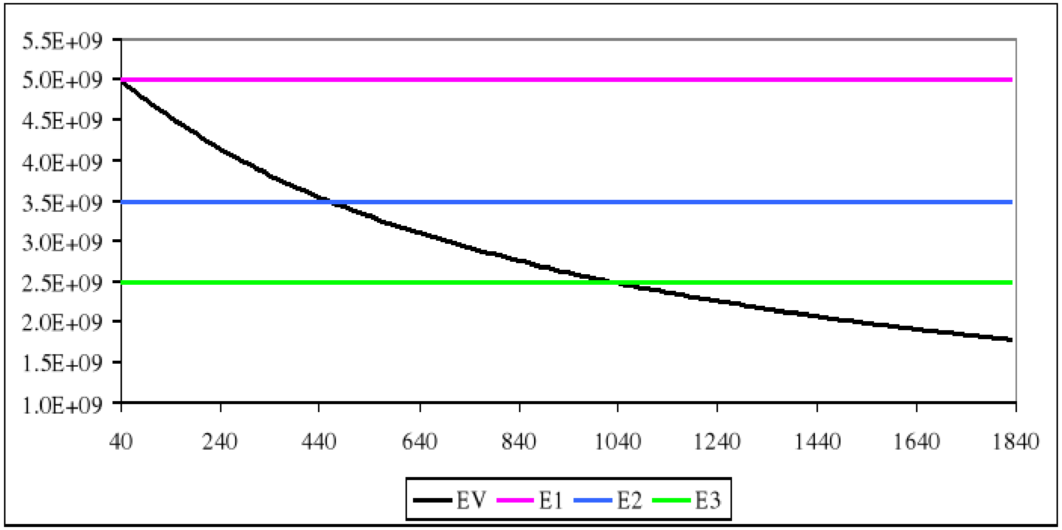
where $\overline{r}$ is the root of the equation

$$\alpha E a b(\theta_a - \theta_b)r^4 - 2p_b b^2(b - a)r^3 + 3a^3[p_a(b - a) - \alpha E b(\theta_a - \theta_b)]r^2$$

$$+2\alpha E a^4 b(\theta_a - \theta_b)r - p_b a^3 b^2(b - a) = 0$$

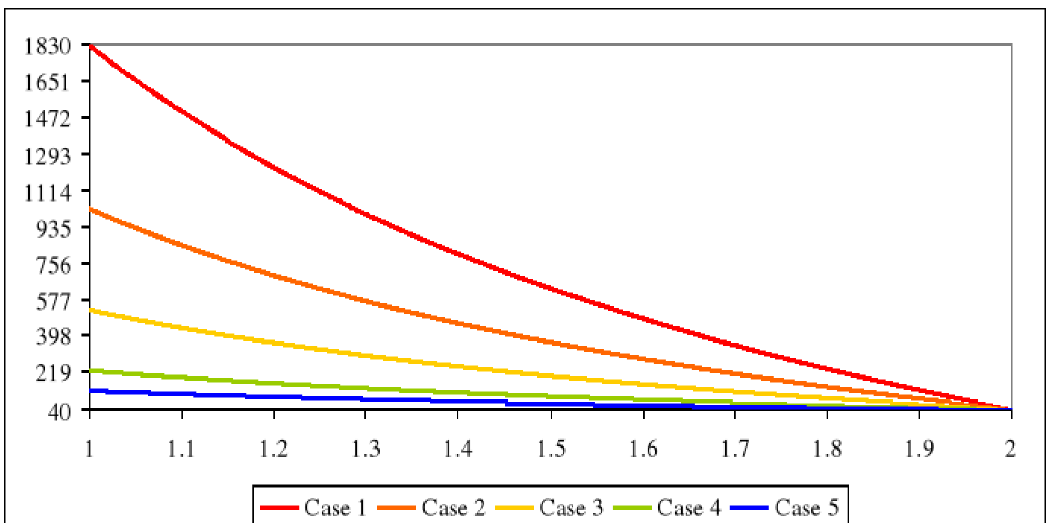
belonging to $[a, b]$ and

$$\overline{\theta} = \theta(\overline{r}), \quad \overline{p} = p_b\frac{b^2}{\overline{r}^2}.$$



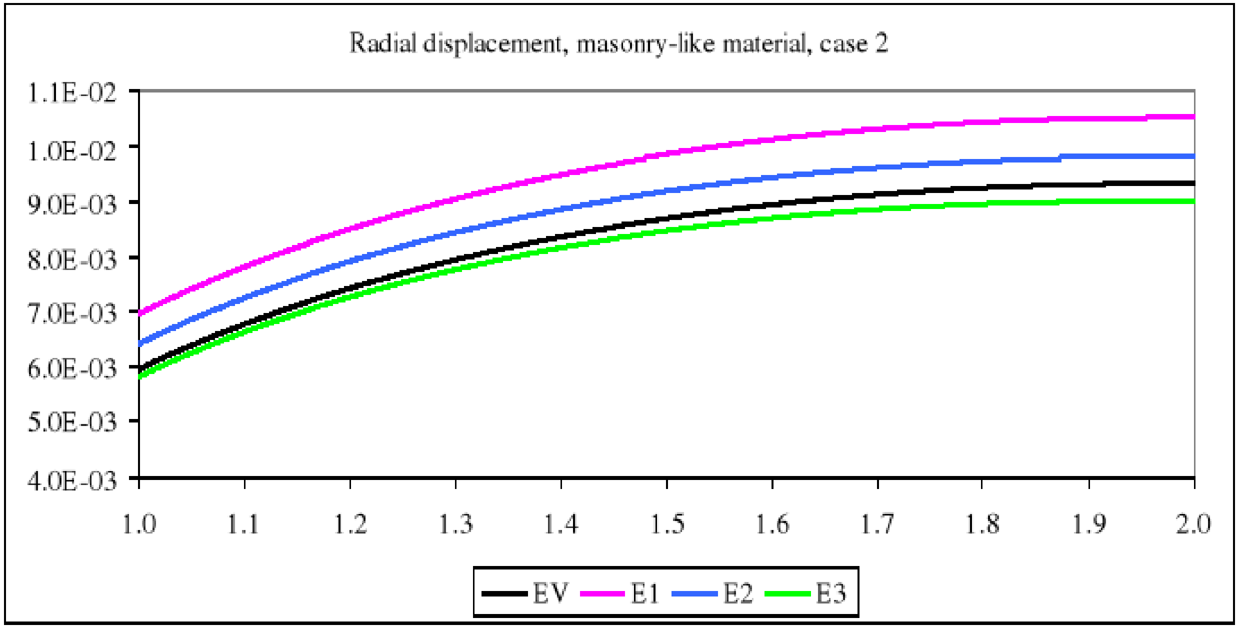
The Young's moduli (Pa) vs. temperature (°C)

$$E_1 = 5000 \text{ MPa},$$
$$E_2 = 3500 \text{ MPa},$$
$$E_3 = 2500 \text{ MPa},$$
$$E_V(\theta) = \frac{4.95\times 10^{12}}{\theta + 950},$$



$$\theta_a = 1830 \text{ °C (case 1)}$$
$$\theta_a = 1030 \text{ °C (case 2)}$$
$$\theta_a = 530 \text{ °C (case 3)}$$
$$\theta_a = 230 \text{ °C (case 4)}$$
$$\theta_a = 130 \text{ °C (case 5)}$$

Steady temperature distribution θ (° C) vs. radius r (m) for different values of θ_a



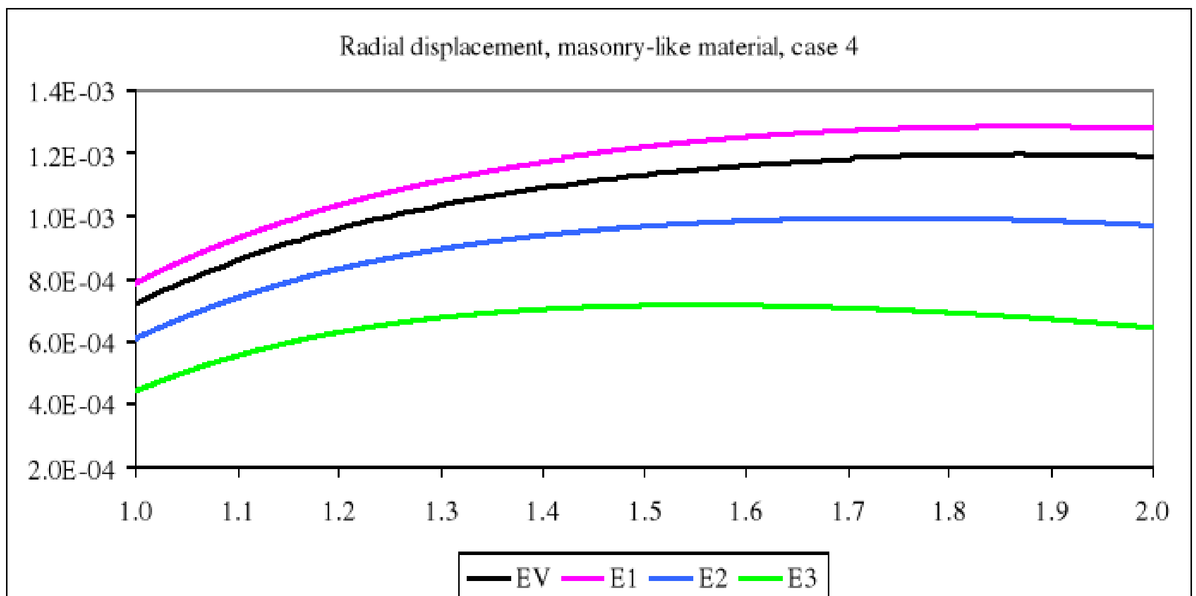
Masonry-like material

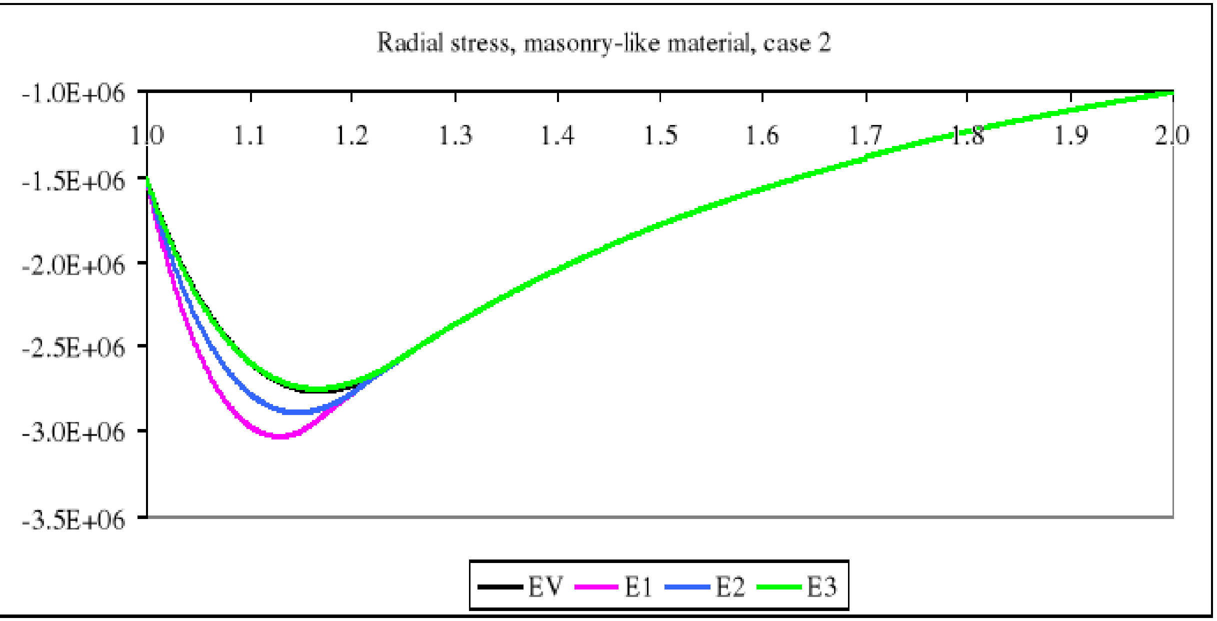
Radial displacement u (m) vs. radius r (m)

$\theta_a = 1030\ °\mathrm{C}$ (case 2)

$\theta_a = 230\ °\mathrm{C}$ (case 4)

$E_1 = 5000$ MPa,
$E_2 = 3500$ MPa,
$E_3 = 2500$ MPa,
$E_V(\theta) = \frac{4.95\times 10^{12}}{\theta + 950}$,





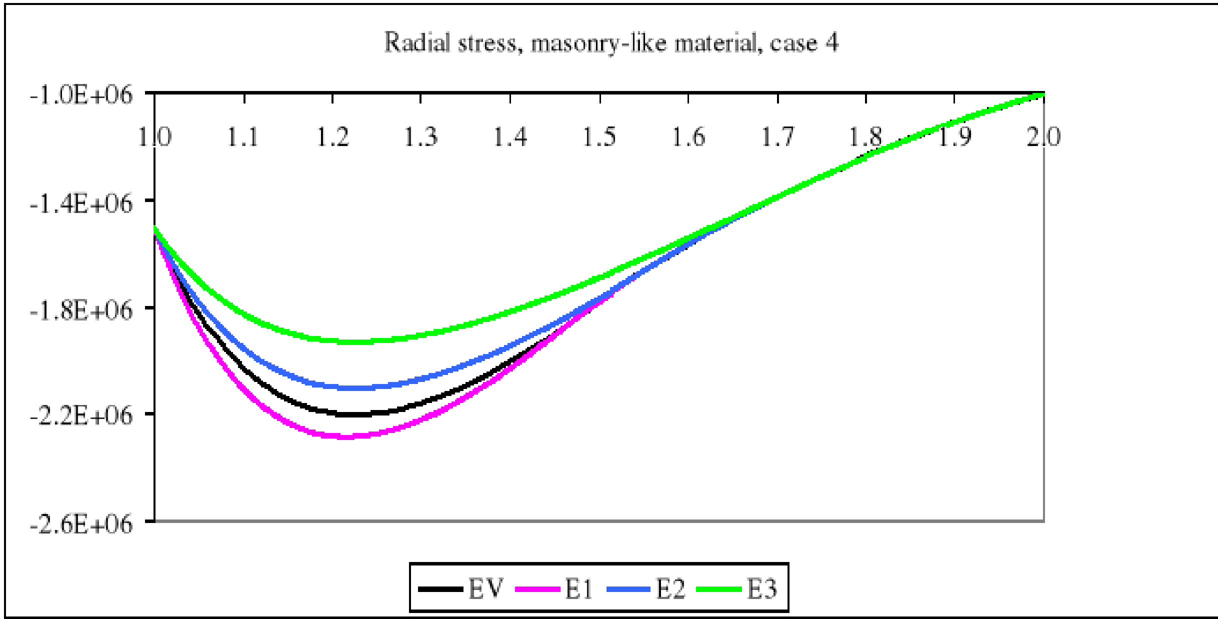
Masonry-like material

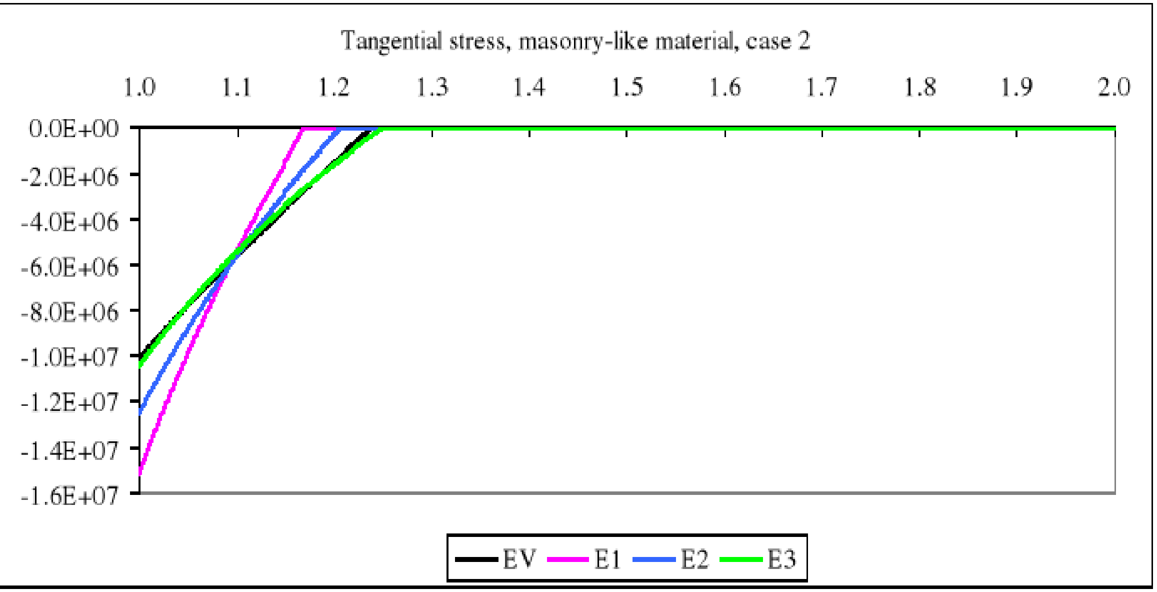
Radial stress σ_r (Pa) vs. radius r (m)

$$\theta_a = 1030\ ^\circ\mathrm{C}\ (\text{case } 2)$$

$$\theta_a = 230\ ^\circ\mathrm{C}\ (\text{case } 4)$$

$$E_1 = 5000\ \mathrm{MPa},$$
$$E_2 = 3500\ \mathrm{MPa},$$
$$E_3 = 2500\ \mathrm{MPa},$$
$$E_V(\theta) = \frac{4.95\times10^{12}}{\theta+950},$$





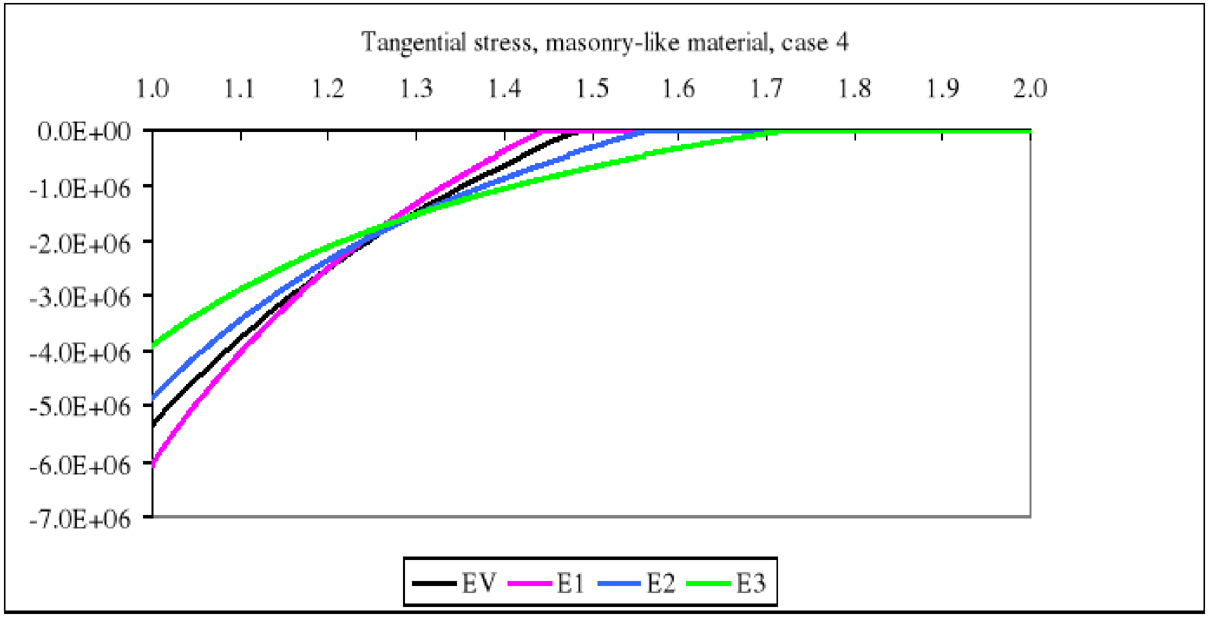
Masonry-like material

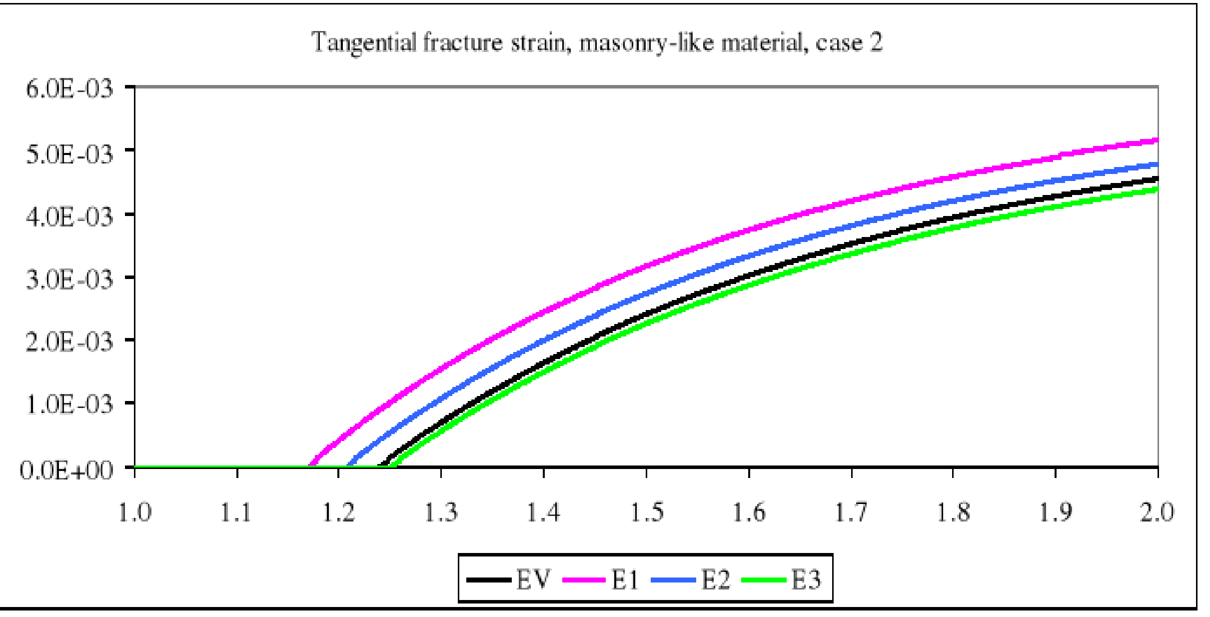
Tangential stress σ_φ (Pa) vs. radius r (m)

$$\theta_a = 1030\ ^\circ\mathrm{C}\ (\text{case } 2)$$

$$\theta_a = 230\ ^\circ\mathrm{C}\ (\text{case } 4)$$

$$E_1 = 5000\ \mathrm{MPa},$$
$$E_2 = 3500\ \mathrm{MPa},$$
$$E_3 = 2500\ \mathrm{MPa},$$
$$E_V(\theta) = \frac{4.95\times 10^{12}}{\theta+950},$$





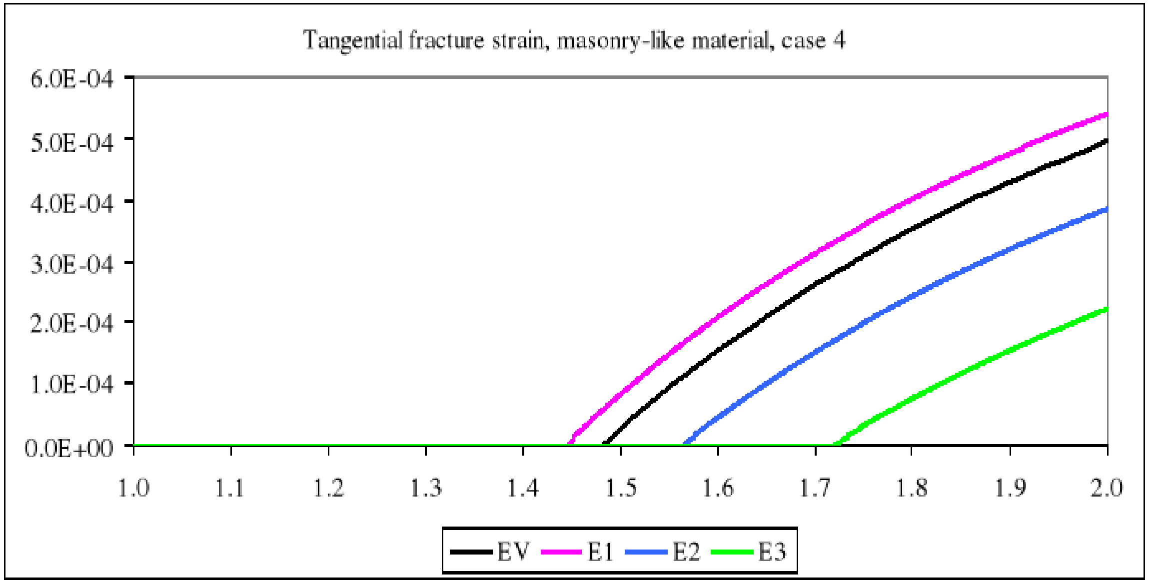
Masonry-like material

Tangential fracture strain $\varepsilon_\varphi^{\mathrm{f}}$ vs. radius r (m)

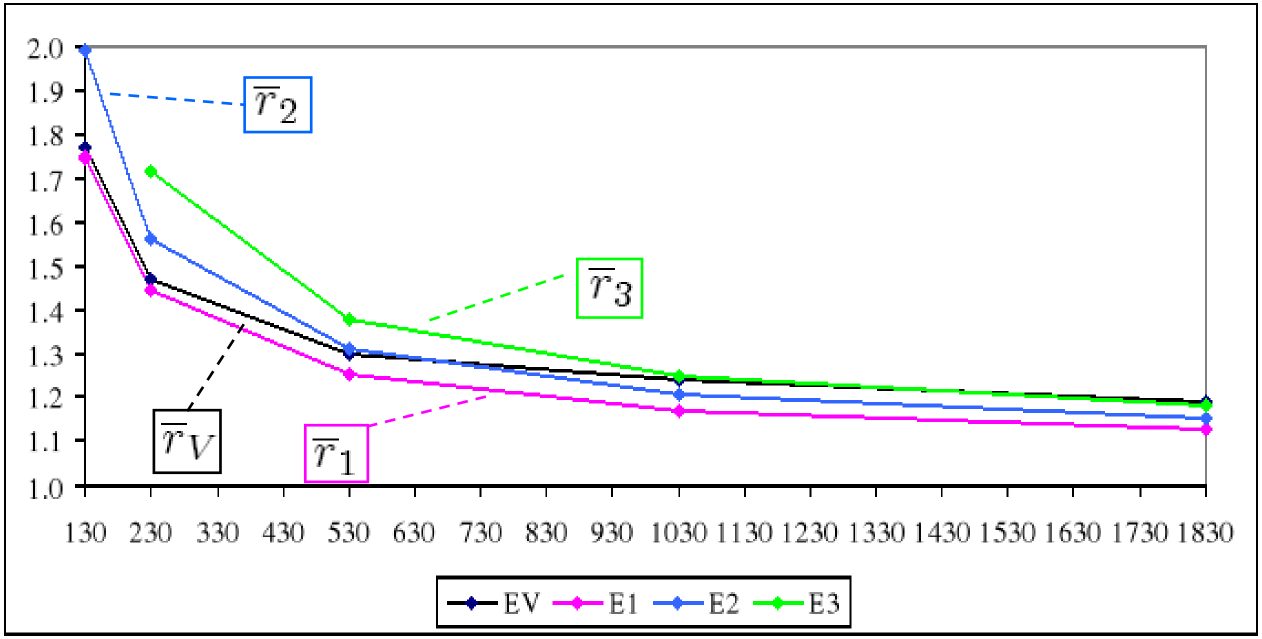
$\theta_a = 1030\ °\mathrm{C}$ (case 2)

$\theta_a = 230\ °\mathrm{C}$ (case 4)

$E_1 = 5000$ MPa,
$E_2 = 3500$ MPa,
$E_3 = 2500$ MPa,
$E_V(\theta) = \frac{4.95\times10^{12}}{\theta+950}$,



Transition radius $\overline{r}$ (m) vs. temperature θ_a (°C)



$E_1 = 5000$ MPa,
$E_2 = 3500$ MPa,
$E_3 = 2500$ MPa,
$E_V(\theta) = \frac{4.95\times10^{12}}{\theta+950}$,

$\theta_a = 1830$ °C (case 1)
$\theta_a = 1030$ °C (case 2)
$\theta_a = 530$ °C (case 3)
$\theta_a = 230$ °C (case 4)
$\theta_a = 130$ °C (case 5)

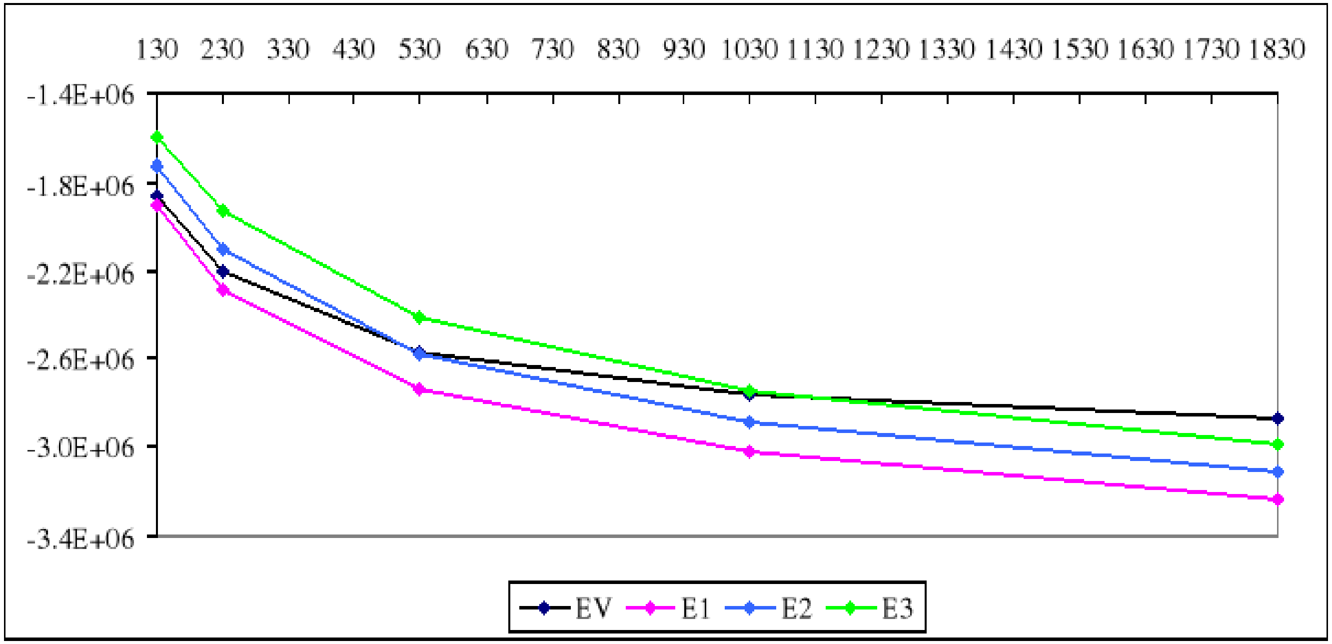
For low values of θ_a ($\theta_a \leq 230$ °C) the radius $\overline{r}$ depends heavily on the the Young's modulus, the lower is E the greater is the entirely compressed region; for $\theta_a = 130$ °C and $E = E_3$, the container is entirely compressed. For $\theta_a \geq 530$ °C, the extension of the cracked region increases and $\overline{r}$ is still a decreasing function of E, even if the values of $\overline{r}$ corresponding to $\theta_a = 1830$ °C are quite close.

Let $\overline{r}_V$, $\overline{r}_1$, $\overline{r}_2$ and $\overline{r}_3$ denote the transition radii corresponding to E_V, E_1, E_2 and E_3, respectively.

$\overline{r}_V$ is close to $\overline{r}_1$ for $\theta_a = 130$ °C and $\theta_a = 230$ °C
$\overline{r}_V$ is close to $\overline{r}_2$ for $\theta_a = 530$ °C
$\overline{r}_V$ is close to $\overline{r}_3$ for $\theta_a = 1030$ °C

$$\sigma_r^{\min} = \min_{r \in [a,b]} \sigma_r(r)$$

$$E_1 = 5000 \text{ MPa},$$
$$E_2 = 3500 \text{ MPa},$$
$$E_3 = 2500 \text{ MPa},$$
$$E_V(\theta) = \frac{4.95 \times 10^{12}}{\theta + 950},$$



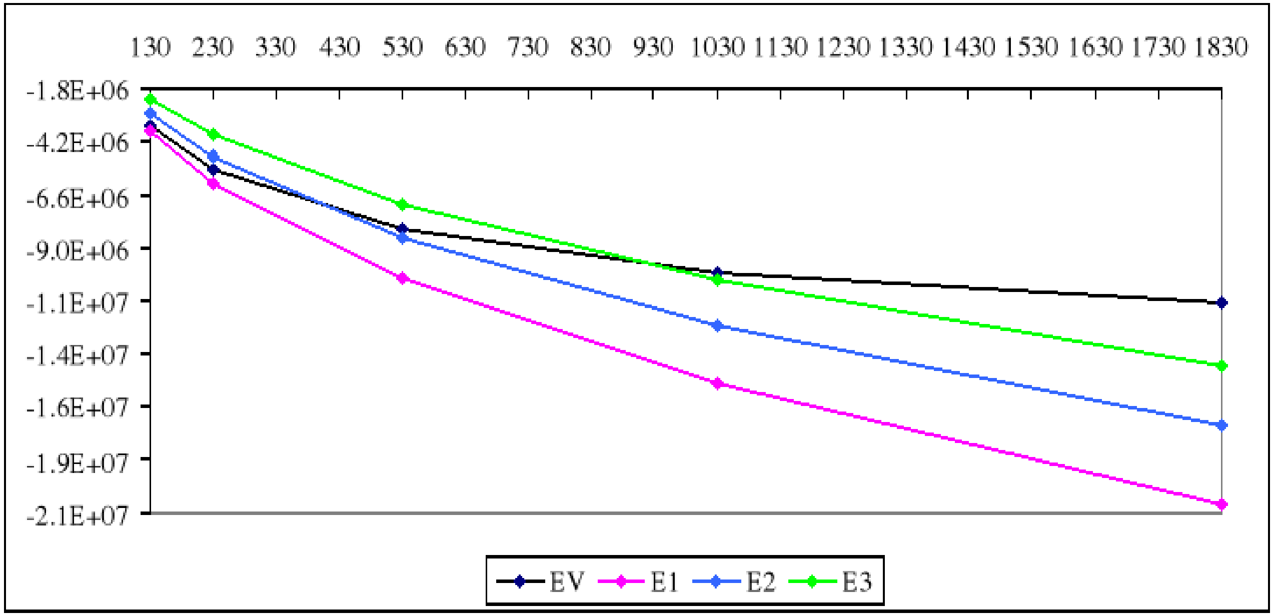
$\sigma_r^{\min}$ (Pa) vs. temperature θ_a (°C)

$$\theta_a = 1830 \text{ °C (case 1)}$$
$$\theta_a = 1030 \text{ °C (case 2)}$$
$$\theta_a = 530 \text{ °C (case 3)}$$
$$\theta_a = 230 \text{ °C (case 4)}$$
$$\theta_a = 130 \text{ °C (case 5)}$$

For $\theta_a \leq 1030$ °C, $\sigma_r^{\min}$ behaves like $\overline{r}$

for $\theta_a = 1830$ °C the absolute values of $\sigma_r^{\min}$ corresponding to E_1, E_2 and E_3 are higher than those obtained with temperature-dependent E_V

$$\sigma_\varphi^{\min} = \min_{r\in[a,b]} \sigma_\varphi(r)$$



$E_1 = 5000$ MPa,
$E_2 = 3500$ MPa,
$E_3 = 2500$ MPa,
$E_V(\theta) = \frac{4.95\times10^{12}}{\theta+950}$,

$\theta_a = 1830$ °C (case 1)
$\theta_a = 1030$ °C (case 2)
$\theta_a = 530$ °C (case 3)
$\theta_a = 230$ °C (case 4)
$\theta_a = 130$ °C (case 5)

$\sigma_\varphi^{\min}$ (Pa) vs. temperature θ_a (°C)

For $\theta_a \leq 1030$ °C, $\sigma_\varphi^{\min}$ behaves like $\overline{r}$

for $\theta_a = 1830$ °C the absolute values of $\sigma_\varphi^{\min}$ corresponding to E_1, E_2 and E_3 are higher than those obtained with temperature-dependent E_V

Conclusions

In order to model numerically the behaviour of masonry-like solids in the presence of thermal variations it is fundamental

1. To have realistic constitutive equations for the material
2. To know the dependence of the material constants on temperature (experimental data)

Further investigation is necessary to assess how temperature-dependent material properties influence the stress field and the crack distribution in the case of thermomechanical coupling